Capture into resonance in dynamics of a classical hydrogen atom in an oscillating electric field

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We consider a classical hydrogen atom in a linearly polarized electric field of slowly changing frequency. In the process of evolution, the system passes through resonances between the driving frequency and the Keplerian frequency of the electron's motion. At a resonance, a capture into the resonance can occur. After the capture, the system evolves in such a way that the resonance condition is approximately preserved, and parameters of the electron's orbit are varying. We study this phenomenon in the case of 2:1 resonance and show that the capture results in growth of the eccentricity of the electron's orbit. It strongly depends on the initial conditions, whether the capture occurs or not. Hence, the capture can be considered as a probabilistic phenomenon. The capture probability is defined and calculated.

I. INTRODUCTION

Dynamics of highly excited (Rydberg) atoms in microwave fields has been a subject of extensive research during the past thirty years. After experiments of Bayfield and Koch [1] and theoretical work of Leopold and Percival [2], it was realized that certain essential properties of the dynamics of Rydberg hydrogen atoms can be described in the frames of classical approach. In more recent studies, methods of classical mechanics yielded surprisingly accurate results for various problems of the hydrogen atom in weak, slowly varying external fields (see [3] for references). In [3], it was shown that this accuracy is due to the fact that in the perturbative limit certain quantum expectation values obey the same equations as the corresponding classical values averaged over the Kepler motion.

One of the major areas of application of classical mechanics is the chaotic ionization of Rydberg atoms. At large enough amplitude of the external field the Chirikov resonance overlap criterion [4] is met, and the system can diffuse in the phase space until the atom is ionized. This way of ionization was thoroughly studied for various configurations and polarizations of external fields. See, for example, ionization in linearly polarized (LP) microwave field [5,6], in circularly polarized (CP) field [7] and references therein, in elliptically polarized (EP) field [8], in CP microwave and magnetic field [9].

Another classical idea in the field is to control the Keplerian motion of the electron using its resonant interaction with a low-amplitude wave of slowly changing frequency. Dynamical problems of this kind were studied in [10] for a 1D model and [11] for a 3D model. In particular, in the latter work an hydrogen atom in a linearly polarized electric field of slowly decreasing frequency was considered. It was shown that at a passage through 2:1 resonance (i.e., when the driving frequency is twice as large as the Keplerian frequency) the system with initially zero eccentricity of the electron's orbit is captured into the resonance. In the cap-

tured state, the electron's Keplerian frequency varies in such a way that the resonant condition is approximately satisfied. In this motion the orbit's eccentricity grows, which may result in ionization of the atom.

In the present work we also consider a 3D hydrogen atom in a linearly polarized electrostatic field of slowly changing frequency. We study behaviour of the system near 2:1 resonance using methods of the theory of resonant phenomena, developed in [12–14] (see also [15]). These methods were previously used in studies of various physical problems including surfatron acceleration of charged particles in magnetic field and electromagnetic wave [16], slowly perturbed billiards [17,18], and classical dynamics in a molecular hydrogen ion [19]. Recently they were also applied to study dynamics of a classical 2D hydrogen atom in a CP microwave field of slowly changing frequency [20].

In the present paper, we show that capture into the resonance necessarily occurs not only in the case of zero initial eccentricity but also if the initial eccentricity is not zero but small enough. Moreover, at larger values of the initial eccentricity the capture is also possible. Following the general approach, capture into the resonance in this latter case can be considered as a probabilistic phenomenon. We define and evaluate its probability. The obtained results can be used to broaden the applicability of the resonant control methods for Rydberg atoms.

The paper is organized as follows. In Sec. II, we use standard techniques of classical celestial mechanics and the theory of resonant phenomena to reduce the equations of motion near the resonance to the standard form. Like in [11], we consider the case of small initial eccentricity. In Sec. III, we apply relevant results of [12] and find the region of so-called "automatic capture" into the resonance at small eccentricities and calculate probability of capture at larger values of initial eccentricity. The capture significantly changes the eccentricity of the electron's orbit and may lead to ionization. In Conclusions, we summarize the results.

II. EQUATIONS OF MOTION NEAR THE 2:1 RESONANCE

We study dynamics of a classical electron in a hydrogen atom perturbed by a harmonically oscillating electric field of

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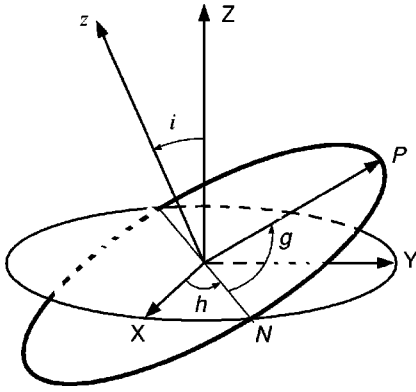*Electronic address: aneishta@iki.rssi.ru, valex@iki.rssi.ru

FIG. 1. The Keplerian ellipse is shown with the bold line. The rest system of coordinates is (XYZ); the z axis is orthogonal to the plane of the orbit. The periapsis and the ascending node are denoted with P and N accordingly.

small amplitude μ, linearly polarized along the Z axis. This system is described with Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + Z\mu \cos \Psi. \qquad (1)$$

Here $\mathcal{H}_0$ is the unperturbed Hamiltonian of motion in the Coulomb field and Ψ is the perturbation phase. Introduce the perturbation frequency $\omega = d\Psi/dt$. We assume that $\omega = \omega(\varepsilon t)$, $0 < \varepsilon \ll 1$, i.e., that this frequency slowly changes with time. For brevity, we put the electron mass and charge to 1, and use dimensionless variables.

The unperturbed trajectory of the electron is an ellipse with eccentricity e (assumed small), semimajor axis a, and inclination i. It is a well-known fact from celestial mechanics that the so-called Delaunay elements L, G, H, l, g, h provide a set of canonical variables for the system under consideration (see, e.g. [21]). The Delaunay elements can be defined as

$$L = \sqrt{a}, \quad G = \sqrt{a(1 - e^2)}, \quad H = \sqrt{a(1 - e^2)} \cos i. \qquad (2)$$

l is the mean anomaly, g is the argument of the periapsis, and h is the longitude of the ascending node (see Fig. 1).

In these variables, Hamiltonian (1) takes the form (see [11])

$$\mathcal{H} = \mathcal{H}_0 + \mu \mathcal{H}_1, \quad \mathcal{H}_0 = -\frac{1}{2L^2},$$

$$\mathcal{H}_1 = \sin i \sum_{k=1}^{\infty} \left[\frac{a_k + b_k}{4} [\sin(kl + g - \Psi) + \sin(kl + g + \Psi)] \right.$$
$$\left. + \frac{b_k - a_k}{4} [\sin(kl - g - \Psi) + \sin(kl - g + \Psi)] \right], \qquad (3)$$

where

$$a_k = \frac{2a}{k} \mathcal{J}_k'(ke), \quad b_k = \frac{2a\sqrt{1 - e^2}}{ke} \mathcal{J}_k(ke). \qquad (4)$$

Here $\mathcal{J}_k(\cdot)$ is the Bessel function of integer order k, and $\mathcal{J}_k'(\cdot)$ is its derivative.

In order to avoid possible singularities at $e = 0$, we make a canonical transformation of variables $(L, G, H; l, g, h) \mapsto (P_3, P_2, P_1; Q_3, Q_2, Q_1)$ defined with generating function $P_3(l + g + h) + P_2(g + h) + P_3 h$. The new canonical variables [called Poincaré elements of the first kind] are expressed in terms of the old ones as follows:

$$P_3 = L, \quad Q_3 = l + g + h,$$

$$P_2 = G - L, \quad Q_2 = g + h,$$

$$P_1 = H - G, \quad Q_1 = h. \qquad (5)$$

As the perturbation frequency slowly varies with time, the system passes through resonances with the unperturbed Keplerian frequency $\dot{l}$. Near a resonance, certain terms in expression (3) for $\mathcal{H}_1$ are changing very slowly. Consider a passage through the 2:1 resonance. In this case, after averaging over fast oscillating terms, we obtain the Hamiltonian describing the dynamics near the resonance:

$$\mathcal{H} = -\frac{1}{2P_3^2} + \mu \alpha(P_2, P_3) \sin i \sin(2Q_3 - Q_2 - Q_1 - \Psi), \qquad (6)$$

where we introduced the notation $\alpha(P_2, P_3) = (a_2 + b_2)/4$. In (6), we have used that at small values of eccentricity $a_2 \approx b_2 \approx ea/2$.

The resonance is defined by $2\dot{Q}_3 = \omega(\tau)$, $\tau = \varepsilon t$. It follows from the unperturbed Hamiltonian (3) that $\dot{Q}_3 = 1/P_3^3$. Hence, denoting the value of P_3 at the resonance as P_r, we find

$$P_r = \left(\frac{2}{\omega(\tau)} \right)^{1/3}. \qquad (7)$$

Our next step is to introduce the resonant phase. We do this with the canonical transformation $(P_3, P_2, P_1; Q_3, Q_2, Q_1) \mapsto (J_3, J_2, J_1; \gamma_3, \gamma_2, \gamma_1)$ defined with the generating function

$$W = J_3(2Q_3 - Q_2 - Q_1 - \Psi) + J_2 Q_2 + J_1 Q_1. \qquad (8)$$

For the new canonical variables we have

$$J_3 = P_3/2, \quad \gamma_3 = 2Q_3 - Q_2 - Q_1 - \Psi,$$

$$J_2 = P_2 + P_3/2, \quad \gamma_2 = Q_2,$$

$$J_1 = P_1 + P_3/2, \quad \gamma_1 = Q_1. \qquad (9)$$

The Hamiltonian function takes the form

$$\mathcal{H} = -\frac{1}{8J_3^2} + \mu \alpha(P_2, P_3) \sin i \sin \gamma_3 - \omega(\tau)J_3. \qquad (10)$$

Near the resonant value $P_3 = P_r$ we can expand this expression into series. With the accuracy of order $O(|P_3 - P_r|^3)$ we obtain the following Hamiltonian:

$$\mathcal{F} = -\frac{3}{2}\frac{(P_3 - P_r)^2}{P_r^4} + \mu\tilde{\alpha}(J_2, P_3)\sin i \sin\gamma_3, \quad (11)$$

where canonically conjugated variables are $P_3/2$ and γ_3 and $\tilde{\alpha}$ is function α expressed via J_2, P_3. Introduce notations $P = P_3$, $\varphi = \gamma_3 + 3\pi/2$, $J = J_2$. The Hamiltonian (11) does not contain γ_2, hence J is an integral of the problem. Another integral is J_1, corresponding to the fact that Delaunay element H is an integral of the original system (3). Coefficient $\sin i$ in (11) should be taken at $P = P_r$. We have

$$\sin i|_{P=P_r} = \sqrt{1 - \frac{H^2}{(J + P_r/2)^2}}. \quad (12)$$

The initial value of eccentricity is small, though not necessarily zero. From (4), we have

$$\tilde{\alpha}(J,P) \approx \frac{ae}{4} = \frac{1}{4}P\sqrt{\left(\frac{P}{2} - J\right)\left(J + \frac{3}{2}P\right)}. \quad (13)$$

Small eccentricity implies that $P/2 - J \ll 1$. As the system evolves near the resonance, small variations of P are essential when calculating the term $(P/2 - J)$ in (13) and less important in the other terms. Hence, in these latter terms, we can put $P = P_r$. We write

$$J + \frac{3}{2}P \approx 2P,$$

where we have used that $P/2 \approx J$. Thus we obtain

$$\tilde{\alpha}(J,P) \approx \frac{\sqrt{2}}{4}P_r^{3/2}\sqrt{\frac{P}{2} - J} \quad (14)$$

and the following expression for the Hamiltonian:

$$\mathcal{F} \approx \mathcal{F}_1 = -\frac{3}{2}\frac{(P - P_r)^2}{P_r^4}$$
$$+ \mu\sqrt{1 - \frac{H^2}{(J + P_r/2)^2}}\frac{\sqrt{2}}{4}P_r^{3/2}\sqrt{\frac{P}{2} - J}\cos\varphi. \quad (15)$$

Introduce so-called Poincaré elements of the second kind:

$$x = \sqrt{P - 2J}\cos\varphi, \quad y = \sqrt{P - 2J}\sin\varphi. \quad (16)$$

The transformation $(P/2, \varphi) \mapsto (x, y)$ is canonical with generating function $W_1 = (y^2/2)\cot\varphi - J\varphi$. Change the sign of $\mathcal{F}_1$ and, to preserve the canonical form, the sign of x. Thus, we obtain the Hamiltonian in the form

$$\mathcal{F}_1 = \frac{3}{2}\frac{(x^2 + y^2)^2}{P_r^4} - 3\frac{x^2 + y^2}{P_r^4}(P_r - 2J) + \mu A(H, J, P_r)x,$$
$$(17)$$

where

$$A(H, J, P_r) = \frac{P_r^{3/2}}{4}\sqrt{1 - \frac{H^2}{(J + P_r/2)^2}}.$$

Note that if the eccentricity is small, then $P_r - 2J \ll 1$, and hence the slow variation of P_r is essential only in the second term in (17). In the other terms, P_r can be assumed to be
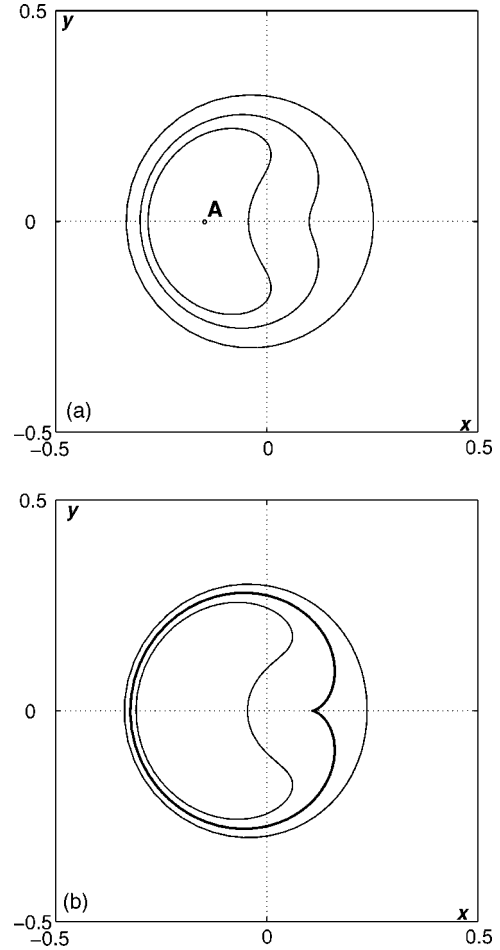

FIG. 2. Phase portraits of the system at fixed values of λ; $\tilde{\mu} = 0.01$. (a) $\lambda = 0.05 < \lambda_*$; (b) $\lambda = 0.069624 \approx \lambda_*$.

constant, say, $P_r = P_{r0}$, where P_{r0} is the value of P_r at $\tau = 0$. Now, we renormalize the Hamiltonian $\mathcal{F}_1 \rightarrow \frac{2}{3}P_{r0}^4\mathcal{F}_1$ to transform it to the following standard form studied in [12]:

$$\mathcal{F}_1 = (x^2 + y^2)^2 - \lambda(x^2 + y^2) + \tilde{\mu}x, \quad (18)$$

with $\lambda = 2(P_r - 2J)$ and $\tilde{\mu} = \mu A(H, J, P_{r0})$. We describe the dynamics defined by Hamiltonian (18) in Sec. III.

III. CAPTURE INTO THE RESONANCE AT SMALL VALUES OF ECCENTRICITY

Dynamics in the system with Hamiltonian function (18) was studied in details in [12]. In this section we put forward the results of [12] relevant to our study.

In (18) $\tilde{\mu} > 0$ is a constant parameter, and λ is a slow function of time, $\dot{\lambda} \sim \varepsilon$. Assume that $\dot{\lambda} > 0$.

On the phase plane (x, y), the values $\sqrt{P - 2J}$ and φ [see (16)] are polar coordinates. Note that eccentricity in the original problem is proportional to $\sqrt{P - 2J}$.

Parameter λ is changing slowly, and as the first step we consider the problem at fixed values of λ. Phase portraits at different values of λ are presented in Figs. 2 and 3. At $\lambda < \lambda_* = \frac{3}{2}\tilde{\mu}^{2/3}$ [Fig. 2(a)] there is one elliptic stable point A on
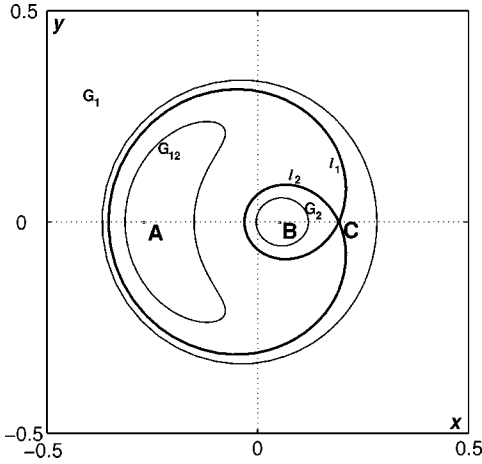
FIG. 3. Phase portrait of the system at $\lambda=0.1>\lambda_*$; $\tilde{\mu}=0.01$.

the portrait. At $\lambda>\lambda_*$ (Fig. 3) there are two elliptic stable points A and B, and one saddle point C. Separatrices l_1,l_2 divide the phase plane into three regions G_1,G_{12},G_2. In Fig. 2(b), the portrait at $\lambda=\lambda_*$ is shown.

The coordinates of point C are $x=x_c$, $y=0$, where $x_c=x_c(\lambda)$ is the largest root of equation

$$\frac{\partial\mathcal{F}_1(x,0,\lambda)}{\partial x}=4x^3-2\lambda x+\tilde{\mu}=0. \qquad (19)$$

At $\lambda\geqslant\lambda_*$, introduce $F_c=F_c(\lambda)=\mathcal{F}_1(x_c,0,\lambda)$ and $\tilde{\mathcal{F}}(x,y,\lambda)=\mathcal{F}(x,y,\lambda)-F_c(\lambda)$. In G_{12} we have $\tilde{\mathcal{F}}<0$, in G_1 and G_2 $\tilde{\mathcal{F}}>0$, on the separatrices $\tilde{\mathcal{F}}=0$.

As parameter λ slowly grows with time, $\dot{\lambda}\sim\varepsilon$, curves l_1, l_2 defined with $\tilde{\mathcal{F}}=0$ slowly move on the phase plane. On time intervals of order ε^{-1} their position on the phase plane essentially changes, together with the areas of G_{12}, G_2. On the other hand, area surrounded by a closed phase trajectory at a frozen value of λ is an approximate integral (adiabatic invariant) of the system with slowly varying parameter λ. Therefore, a phase point can cross l_1, l_2 leaving one of the regions G_i and entering another region.

Denote with $[x(t),y(t)]$ a phase point moving according to (18). Without loss of generality assume that $\lambda=\lambda_*$ at $t=0$. The initial point $(x(0),y(0))$ can be either inside l_1 [$\tilde{\mathcal{F}}_0=\tilde{\mathcal{F}}(x(0),y(0),\lambda_*)<0$] or out of l_1 [$\tilde{\mathcal{F}}_0>0$]. The following assertion is valid. All points lying inside l_1 at $\lambda=\lambda_*$ except, maybe, of those belonging to a narrow strip $-k_1\varepsilon^{6/5}\leqslant\tilde{\mathcal{F}}_0<0$, where k_1 is a positive constant, stay in G_{12} at least during time intervals of order ε^{-1}.

This result is due to the fact that the area of G_{12} monotonically grows with time [see below (22) and (23)], and conservation of the adiabatic invariant makes a phase point go deeper and deeper into this region. A point captured in G_{12} rotates around point A$=(x_A,0)$, where x_A is the smallest root of Eq. (19). As time grows, $|x_A|$ also grows and point A on the portrait slowly moves along x axis in the negative direction. Therefore, the motion is a composition of two components: fast rotation along a banana-shaped curve sur-

rounding A and slow drift along x axis. The area surrounded by each banana-shaped turn is approximately the same and equals the area $\mathcal{S}(\lambda_*)$ surrounded by the trajectory passing through $(x(0),y(0))$ at $\lambda=\lambda_*$. Hence, the average distance between the phase point and the origin slowly grows, corresponding to the eccentricity growth in the original problem.

In [11], it was shown that a point having zero initial eccentricity necessarily undergoes the eccentricity growth. The formulated result implies that this is also valid for all the points initially [i.e., at $\lambda=\lambda_*$] inside l_1, except, maybe, for a narrow strip close to l_1. A typical linear size of this domain is of order $\tilde{\mu}^{1/3}$. In [22] this phenomenon was described in the problem of motion of Saturn's satellites and called "automatic entry into libration" (see also [23]).

Consider now the case when the point $(x(0),y(0))$ is outside l_1: $\tilde{\mathcal{F}}_0>0$. With time the area inside l_1 grows, and at a certain moment the phase trajectory crosses l_1. In the adiabatic approximation the area surrounded by the phase trajectory is constant: $\mathcal{S}(\lambda)=\mathcal{S}(\lambda_*)$. Hence, in this approximation the time moment of crossing l_1 can be found from equation $\mathcal{S}_1(\Lambda)=\mathcal{S}(\lambda_*)$. Here $\mathcal{S}_1(\lambda)$ is the area inside l_1 as a function of λ and Λ is a value of the parameter at this moment. After crossing, there are two possibilities: (i) the phase point can continue its motion in G_{12} during a time interval of order at least ε^{-1} (this corresponds to capture into the resonance and growth of the eccentricity); (ii) without making a full turn inside G_{12} the phase point can cross l_2 and continue its motion in G_2 (this corresponds to passage through the resonance without capture). The area of G_2 also monotonically grows with time [see below (22) and (23)], hence such a point cannot cross l_2 once more and return to G_{12}.

It is shown in [12] that the scenario of motion after crossing l_1 strongly depends on initial conditions $(x(0),y(0))$: a small, of order ε, variation of initial conditions can result in qualitatively different evolution. If the initial conditions are defined with a final accuracy δ, $\varepsilon\ll\delta\ll1$, it is impossible to predict *a priori* the scenario of evolution. Therefore, it is reasonable to consider capture into G_{12} or G_2 as random events and introduce their probabilities.

Following [24], consider a circle of small radius δ with the center at the initial point M_0. Then the probability of capture into G_{12} is defined as

$$\mathcal{P}=\lim_{\delta\to0}\lim_{\varepsilon\to0}\frac{S_{12}}{S_{M_0}^{\delta}}, \qquad (20)$$

where $S_{M_0}^{\delta}$ is the measure of the circle of radius δ and S_{12} is the measure of the set of points inside this circle that are captured finally into G_{12}.

Let $\lambda=\Lambda$ be the parameter value at the moment of crossing l_1 in the adiabatic approximation. The following formula for probability P is valid:

$$\mathcal{P}=\frac{I_1-I_2}{I_1}, \quad \text{where } I_1(\lambda)=-\oint_{l_1}\frac{\partial\tilde{\mathcal{F}}}{\partial\lambda}dt,$$

$$I_2(\lambda)=-\oint_{l_2}\frac{\partial\tilde{\mathcal{F}}}{\partial\lambda}dt, \qquad (21)$$

and the integrals I_1, I_2 are calculated at $\lambda=\Lambda$. Calculating the integrals, one finds [12]
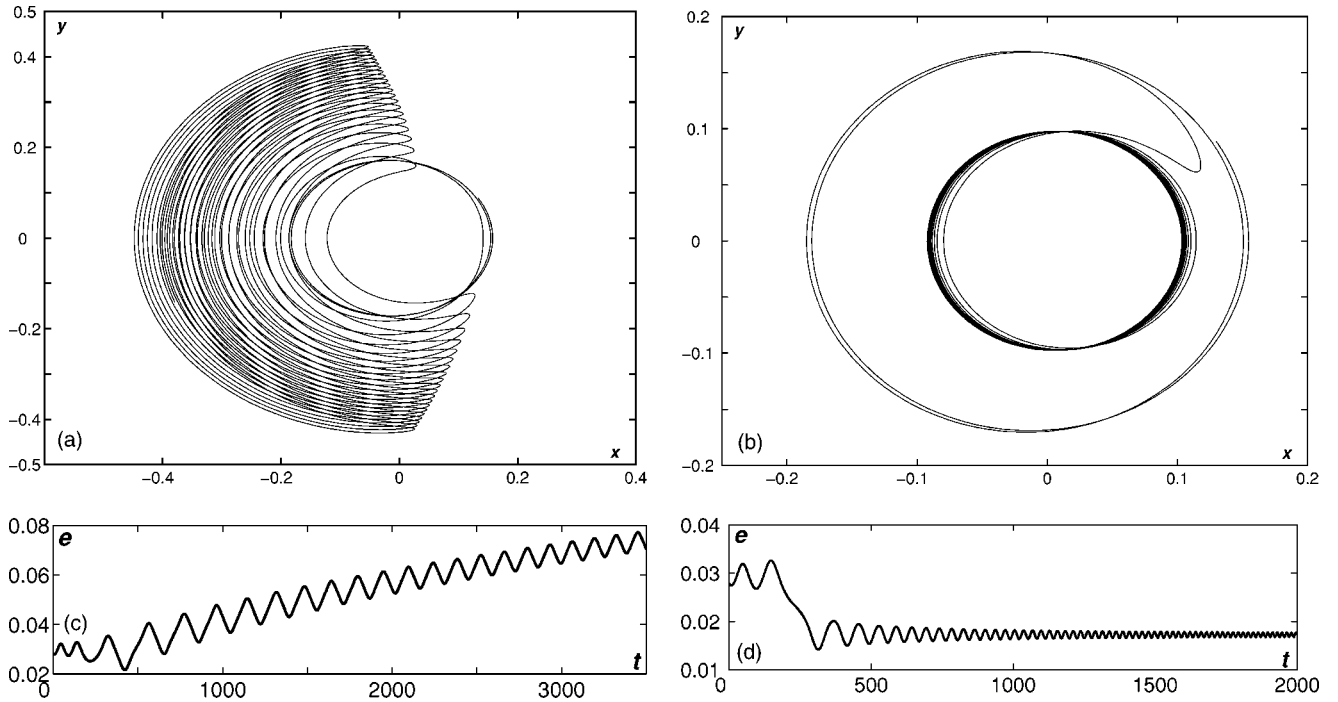
FIG. 4. Passage through the resonance on (x,y) plane and corresponding variation of eccentricity (bottom plots); $\tilde{\mu}=0.001$; $\varepsilon=0.0001$. (a) Passage with capture into the resonance; (b) passage without capture; initial conditions in the cases (a) and (b) are different.

$$I_1(\Lambda) = \frac{1}{2}(2\pi - \Theta), \quad I_2 = \frac{\Theta}{2}, \quad \Theta = \arccos\left(\frac{\Lambda}{2x_c^2} - 2\right).$$
(22)

Here Θ is the angle between the tangencies to l_1 at C, $0 \leq \Theta < \pi$. Note that

$$\frac{d\mathcal{S}_2}{d\lambda} = I_2, \quad \frac{d\mathcal{S}_{12}}{d\lambda} = I_1 - I_2.$$
(23)

Geometrically, formula (21) can be interpreted as follows. In a Hamiltonian system, phase volume is invariant. As parameter λ changes by $\Delta\lambda$, a phase volume ΔV_{12} enters the region G_{12}. At the same time, a volume ΔV_2 leaves this region and enters G_2. The relative measure of points captured in G_{12} is $(\Delta V_{12} - \Delta V_2)/\Delta V_2$. The integral I_1 in (21) is the flow of the phase volume across l_1, and I_2 is the flow across l_2. Therefore, $\mathcal{P}$ gives the relative measure of points captured into G_{12}.

Note that, rigorously speaking, there also exists a set of initial conditions that should be excluded from consideration. Phase trajectories with initial conditions in this set pass very close to saddle point C, and the asymptotic results of [12] cannot be applied to them. However, this exclusive set is small: it can be shown that its relative measure tends to zero faster than any positive power of ε.

In Fig. 4(a), capture into the resonance is shown. First, the phase point rotates around the origin in region G_1, then it crosses l_1, enters region G_{12} and continues its motion in this region. In the course of this motion, the average distance from the origin grows, corresponding to the growth of the eccentricity. The motion of captured phase points is regular

in the sense that two initially close captured phase points stay close to each other in the process of further evolution. In Fig. 4(b), all the parameter values are the same as in Fig. 4(a), but initial conditions are slightly different. In this case, after crossing l_1, the phase point crosses l_2 and gets into region G_2. This is a passage through the resonance without capture. Variation of eccentricity in this case is due to transition from G_1 into G_2. For numerics, we used $\lambda = \lambda_0 + \varepsilon t$.

We have tested formula (21) for probability of capture numerically. We considered a square of size $100\varepsilon \times 100\varepsilon$ on (x,y) plane centered at the point $(\bar{x}_0, \bar{y}_0)$ where the formula is tested, and took 100 initial points for the system with Hamiltonian (18) in this square. The numerical value of capture probability at the central point $\mathcal{P}_{num}$ is the ratio of captured trajectories to the total number of trajectories. The results of numerical test show considerably good agreement with the formula. A part of these results for $\tilde{\mu}=0.001$, $\varepsilon=0.0001$ are presented in Table I.

At the end of this section, we comment on the parameters' range appropriate for the methods used. First of all, it must be $\mu \ll 1$ and $\varepsilon \ll 1$, otherwise it is not possible to use the

TABLE I. Numerical value of capture probability $\mathcal{P}_{num}$ vs theoretical value $\mathcal{P}_{th}$.

$\bar{x}_0$	$\bar{y}_0$	$\mathcal{P}_{num}$	$\mathcal{P}_{th}$
0.255	0.105	0.14	0.146
0.225	0.115	0.12	0.130
0.275	0.085	0.10	0.120
0.295	0.115	0.09	0.101
0.305	0.125	0.11	0.096

perturbation theory methods. One can also give some additional restrictions. Consider Hamiltonian (18). A typical scale on the corresponding phase portrait can be found using the condition that the first and the last terms in the Hamiltonian are of the same order. Thus, we find that typical values of x and y on the portrait are of order of $\mu^{1/3}$. Hence we find that typical frequency of motion on this portrait is $\Omega_{typ} \sim \dot{\phi} \sim \mu^{2/3}$. The adiabaticity condition (i.e., that variation of the driving frequency during a period of motion is much smaller than the frequency of motion) can be written as $\varepsilon \ll \Omega_{typ}^2$ or $\varepsilon \ll \mu^{4/3}$. Therefore, of the two small parameters, ε should always be the smallest. Also, it is necessary for application of the time averaging that the driving frequency is much larger than the typical frequency of motion on the phase portrait: $\omega \gg \Omega_{typ}$. Thus, we obtain $\mu \ll \omega^{3/2}$. Essentially, this condition provides possibility of using the isolated resonance approximation.

IV. CONCLUSIONS

Summarizing the results, we can say the following. (i) Capture into the resonance in the considered problem results in growth of the eccentricity of the electron's orbit. (ii) On the phase plane around the origin ($e=0$), there exists a region of size of order $\tilde{\mu}^{1/3}$ such that all phase trajectories with initial conditions (i.e., at $\lambda = \lambda_*$) in this region undergo a capture into the resonance with necessity ("automatic capture"). (iii) If initial eccentricity is larger, and the initial point on the phase plane is out of the region mentioned above, there is a finite probability that the phase trajectory will be captured into the resonance. This probability is given by (21).

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